G/G models as the strong coupling limit of topologically massive gauge theory

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Abstract

We show that the problem of computing the vacuum expectation values of gauge invariant operators in the strong coupling limit of topologically massive gauge theory is equivalent to the problem of computing similar operators in the G_k/G model where k is the integer coefficient of the Chern-Simons term. The G_k/G model is a topological field theory and many correlators can be computed analytically. We also show that the effective action for the Polyakov loop operator and static modes of the gauge fields of the strongly coupled theory at finite temperature is a perturbed, non-topological G_k/G model. In this model, we compute the one loop effective potential for the Polyakov loop operators and explicitly construct the low-lying excited states. In the strong coupling limit the theory is in a deconfined phase.

1   Introduction

Chern-Simons theory [1, 2] is a three dimensional topological quantum field theory whose observables and correlation functions are intimately related the topology of three dimensional manifolds [3]. It has found physical applications to quasi 2+1-dimensional systems where some of the topological effects which are associated with it are considered important. A well known example is the use of both abelian and non-abelian Chern-Simons theory to describe the the quantum Hall states. There, the exotic statistics of quasiparticles [4] which arise from their coupling to Chern-Simons theory is an essential feature. Another example are speculations about the mechanism for high temperature superconductivity [5]. A very interesting recent application is the relation of Chern-Simons theory to quark-gluon plasmas in four dimensions [6].

The Chern-Simons action is the volume integral of a three-form

$$S_{CS} = \frac{k}{4\pi} \int d^3x \, \text{Tr} \left(AdA - \frac{2}{3}iA^3 \right) \tag{1}$$

where A is the connection one-form and k is an integer. This action involves no dimensional parameters. Furthermore, being the volume integral of a three-form, it does not contain the metric of the three dimensional space and it is therefore invariant under general coordinate transformations. This is a characteristic feature of a topological field theory. Assuming that the integration measure can be appropriately defined, the partition function

$$Z = \int [DA] \, e^{iS_{CS}[A]} \tag{2}$$

is independent of local geometry and depends only on topological data.

In physical applications, coupling of the gauge field to other degrees of freedom of a physical system usually breaks the general coordinate invariance of Chern-Simons theory so that it is no longer a topological field theory. Generally, the leading correction to the effective action for the gauge field in powers of gauge fields and their derivatives is a Yang-Mills term,[1]

$$S_{YM} = - \int d^3x \, \frac{1}{2e^2} \text{Tr} \left(F_{\mu\nu} F^{\mu\nu} \right) \tag{3}$$

where F is the field strength and e^2 is a parameter with the dimension of mass. Such a term would be generated by radiative corrections from relativistic matter fields with a mass gap. The coupling constant e^2 is typically of the same magnitude as the mass gap. In this Paper, we shall consider this theory in the limit where the dimensional constant e^2 is large compared to other momentum scales of interest.

If the matter fields which the gauge field coupled were not relativistically invariant, but had a non-relativistic, spatially isotropic spectrum, the induced Yang-Mills action would have two independent parameters,

$$S_{YM}^{\text{nr}} = - \int d^3x \, \text{Tr} \left(\frac{1}{2e^2} F_{0i} F^{0i} + \frac{1}{2P_M} F_{ij} F^{ij} \right) \tag{4}$$

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[1]For certain kinds of matter fields, there could also be renormalization of the Chern-Simons term [7].

with e^2 being related to the electric permeability and P_M the magnetic permittivity. The strong coupling limit which we shall consider in this paper applies to the situation where

$$1/e^2 >> 1/P_M \qquad (5)$$

or where P_M is much larger than any other dimensional parameters of interest, so that the electric term dominates and the magnetic term can be ignored entirely. This is typical in the case where the average velocities and effective charges of the non-relativistic matter are small so that their motions are unaffected by magnetic fields. If (5) holds, the analysis which we shall describe in this paper is valid without the constraint that e^2 also be large. In the following, for the purposes of discussion, we shall assume that the Yang-Mills action has the relativistic form (3) and take the strong coupling, large e^2 limit.

The presence of the Yang-Mills term in the action breaks the general coordinate invariance of the Chern-Simons term explicitly. It also renders the gauge interaction super-renormalizable. It could be introduced for that purpose, as a higher derivative ultraviolet regulator. In that case e^2 would be of order the ultraviolet cutoff.

Topologically massive Yang-Mills theory, which is the model with both the Chern Simons and Yang Mills terms in the action, [1, 2],

$$S = S_{YM} + S_{CS} \qquad (6)$$

describes a self-interacting massive vector field in 2+1-dimensions. The coupling is weak when the integer k is large, which also means that the vector field mass $\mu = e^2 k/4\pi$ is large.

In this Paper, we shall make some observations concerning the strong coupling limit, $e^2 \to \infty$, with k fixed and not necessarily large, of topologically massive gauge theory[2]. That limit should recover Chern-Simons theory with some integer parameter k' (not necessarily the original classical k). We shall show that, in the Hamiltonian formalism, it is also equivalent to another type of topological field theory, the G/G model, which is also known (by other means [8, 9, 10, 11]) to be related to Chern-Simons theory.

Typically, the strong coupling limit of a gauge theory is not a renormalizable field theory. It is interesting that topologically massive gauge theory avoids this fate by making use of general coordinate invariance, i.e. that the strongly coupled theory itself is a topological field theory. From a physical point of view, in the strong coupling limit, the mass gap of the theory is taken to infinity, leaving only dynamics of the vacuum which involves no propagating field theoretical degrees of freedom which could produce ultraviolet divergences.

The resulting quantum mechanical degrees of freedom are described by the G/G model where, in principle, many interesting correlators can be computed exactly. We shall show that the problem of computing the ground state expectation value of a wide array of static gauge invariant operators in the strong coupling limit of

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[2]Since e^2 is a dimensionful parameter, what is meant by this limit is that e^2 should be large compared to all momentum scales of interest. Later at finite temperature, the appropriate limit will be that $e^2 T \to \infty$ in such a way that e^2/T remains finite.

2+1-dimensional topologically massive gauge theory is equivalent to the problem of computing similar observables in the G_k/G model where k is the integer quantized coefficient of the Chern-Simons term in the topologically massive gauge theory.

The G_k/G models [8, 9, 10, 11, 12, 13, 14] themselves are an interesting class of topological field theories related to two dimensional Yang-Mills theory. They are obtained by taking a conformal field theory, the Wess-Zumino-Novikov-Witten (WZNW) model [15, 16], whose dynamical variables are loop group elements, $g(x) \in G$, and introducing gauge fields so that the model is symmetric under the gauge transformation

$$g(x) \to u^{-1}(x)g(x)u(x) \;\;, \;\; A(x) \to u^{-1}(x)A(x)u(x) + iu^{-1}(x)du(x) \qquad (7)$$

The gauge fields are integrated over without a kinetic term, so that they act effectively as Lagrange multipliers. With the gauge field taking values in an anomaly-free subgroup of G, the gauged WZW model reproduces the coset construction of conformal field theories [17]. When the gauge fields take values in whole G, the result is a topological field theory which is also a $c = 0$ conformal field theory. The conformal field theory has been used to compute the Verlinde formula which relates the topology of group theoretical numbers to the dimensions of conformal blocks [10, 11]. Here we shall use the Verlinde formula to show that the center-symmetry of topologically massive gauge theory is spontaneously broken in infinite volume, consistent with the deconfined nature that one would expect for a topologically massive gauge theory.

A more complex issue is to understand the nature of excited states in the strong coupling limit. In a conventional gauge theory, 3+1-dimensional Yang-Mills theory for example, in the strong coupling limit, excited states contain static, infinitely massive strings of electric flux. In a topologically massive gauge theory, they turn out to be quite different objects. The spectrum is similar to a tower of Landau levels, with excited states being occupation of the higher levels. Here, we will consider their contribution to thermodynamic states by constructing an effective field theory for the finite temperature strong coupling model. Indeed, to have nontrivial thermodynamics, the temperature will be taken as large enough to excite some of the states of the underlying theory. That is, the finite temperature theory is no longer Chern-Simons theory but the true strong coupling limit of topologically massive Yang-Mills theory where some of the massive degrees of freedom are excited. We shall find that the partition function is described by a non-renormalizable effective field theory, which is a perturbed G/G model. The degrees of freedom in this effective model are the static, magnetic modes of the vector gauge field and the group valued field whose trace is the Polyakov loop operator.

This model is a topological field theory at the classical level, but the general coordinate invariance is broken by anomalies when the model is quantized. In fact, for arbitrary parameters it is non-renormalizable. However, we will find that, when it is tuned so that it is logarithmically close to a topological field theory, that is in the limit where the dimensionless ratio of coupling constant to temperature is tuned as $e^2/T \sim \ln \Lambda/\mu$ (with Λ the ultraviolet cutoff) the effective potential for the

group valued field can be made finite to one loop. In this limit, we then have some indication that the theory is one-loop renormalizable. Then, in principle, correlators of static operators such as the Polyakov loop operators and moments of the magnetic field can be computed to one-loop order.

We shall also examine the nature of excited states in the strong coupling limit. We shall find that the excited states themselves can be constructed by using holomorphic factorization properties of the G/G model. We will comment on their properties.

In Section 2 we review the quantization of topologically massive Yang-Mills theory in the Hamiltonian picture. The material here is standard and can be found in many places in the literature. Here we follow the original work of Deser, Jackiw and Templeton [2]. In Section 3 we study the strong coupling limit of that model and in Section 4 we show how it is related to G/G models and compute some of the Polyakov loop correlators. In Section 5 we examine the strong coupling model at finite temperature and find that it is given by a perturbed G/G model. In Section 6 we discuss the quantum structure of the perturbed G/G model and show that the effective potential for the gauge group variables can be made finite at one loop. In Section 7 we examine the nature of excited states of the strong coupling theory. We construct non-Abelian Landau levels on punctured Riemann surfaces and show that these states saturate the thermal ensemble of the theory. In Section 8 we summarize the conclusions.

2   Topologically massive gauge theory revisited

Consider a gauge theory in 2+1 spacetime dimensions where the basic dynamical object is the Lie algebra-valued 1-form $A = A_\mu(x)dx^\mu$ whose density $A_\mu(x)$ is the gauge field. The model has gauge symmetry under $A^g = g^{-1}Ag + ig^{-1}dg$ where $g(x)$ is a smooth function on 2+1-dimensional spacetime taking values in the fundamental representation of a compact semisimple Lie group. The action is $S = S_{CS} + S_{YM}$, with S_{CS} and S_{YM} defined in Eqs. (1) and (3) respectively.[3]

We begin by reviewing the canonical quantization of topologically massive gauge theory. We suppose that all fields vanish sufficiently fast at spatial infinity, so that the three-manifold we are quantizing on is of the form $\Sigma \otimes \mathbb{R}$, with Σ the Riemann sphere. The dynamical variables in phase space are the spatial components of the gauge field, $A_i(\vec{x})$, and the associated canonical momenta, $\Pi_i(\vec{x})$. The equation of motion for the temporal component of the gauge field A_0, Gauss' law

$$\mathcal{G} \equiv \frac{1}{e^2}D_iF_{0i} + \frac{k}{4\pi}F_{12} = 0 \ , \tag{8}$$

will be a constraint on the phase space variables. The canonical momentum conjugate to the spatial component of the vector field is

$$\Pi_i^a(\vec{x}) = \frac{\partial}{\partial \dot{A}_i^a(\vec{x})}L = \frac{1}{e^2}F_{0i}^a(\vec{x}) + \frac{k}{8\pi}\varepsilon_{0ij}A_j^a(\vec{x}) \tag{9}$$

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[3] *Conventions*: The covariant derivative is $D_\mu = \partial_\mu - iA_\mu$ and the curvature tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. Hermitean generators of the Lie algebra $[T^a, T^b] = if^{abc}T^c$ are normalized in the fundamental representation as $\text{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. The connection and curvature are $A_\mu = A_\mu^a T^a$, $F_{\mu\nu} = F_{\mu\nu}^a T^a$.

We define the (unconventionally normalized) electric field as

$$E_i(\vec{x}) = \frac{1}{e^2}F_{0i}(\vec{x}) = \Pi_i(\vec{x}) - \frac{k}{8\pi}\varepsilon_{ij}A_j(\vec{x}) \tag{10}$$

and the magnetic field as

$$B(\vec{x}) = F_{12}(\vec{x}) \tag{11}$$

The Hamiltonian is

$$H = \int d^2x \left(e^2 \ \mathrm{Tr} \left(\vec{E}^2(\vec{x}) \right) + \frac{1}{e^2} \ \mathrm{Tr} \left(B^2(\vec{x}) \right) \right) \tag{12}$$

The canonical commutation relation for the gauge field and its canonical momentum is

$$\left[A_i^a(\vec{x}), \Pi_j^b(\vec{y}) \right] = i\delta_{ij}\delta^{ab}\delta(\vec{x}-\vec{y}) \tag{13}$$

The electric field defined in (10) is analogous to the velocity operator and differs from the canonical momentum by $\mathcal{A}_i^a(x;A) = k\varepsilon_{ij}A_j^a(\vec{x})/8\pi$. The commutator of the electric fields is

$$\left[E_i^a(\vec{x}), E_j^b(\vec{y}) \right] = -\frac{ik}{4\pi}\varepsilon_{ij}\delta^{ab}\delta(\vec{x}-\vec{y}) \tag{14}$$

Also

$$\left[A_i^a(\vec{x}), E_j^b(\vec{y}) \right] = i\delta^{ab}\delta_{ij}\delta(\vec{x}-\vec{y}) \tag{15}$$

In terms of phase space variables, the gauge constraint (8) is

$$\mathcal{G}(\vec{x}) = \partial_i E_i(\vec{x}) - i\left[A_i(\vec{x}), E_i(\vec{x}) \right] + \frac{k}{4\pi}B(\vec{x}) \sim 0 \tag{16}$$

The weak equality, $\sim$, in (16) implies that, rather than solving (16) as an operator equation, we will impose it as a physical state condition in the quantum theory. States in the physical sector of the quantum Hilbert space are in the kernel of the constraint operator

$$\mathcal{G}(\vec{x}) \ \Psi_{\mathrm{phys.}} = 0 \tag{17}$$

The operator $\mathcal{G}(\vec{x})$ is the generator of infinitesimal time-independent gauge transformations and commutes with gauge invariant operators, such as the energy density and the Hamiltonian,

$$[\mathcal{G}(\vec{x}), \ H] = 0 \ . \tag{18}$$

In the functional Schrödinger picture, the gauge field operator $\vec{A}(\vec{x})$ is treated as a position variable. The quantum states are wave functionals of gauge field configurations, $\Psi[A]$, and the canonical momentum is the functional derivative operator,

$$\Pi_i^a(\vec{x}) \equiv \frac{1}{i}\frac{\delta}{\delta A_i^a(\vec{x})} \tag{19}$$

The stationary states satisfy the functional Schrödinger equation

$$\int d^2x \left(\frac{e^2}{2}\sum_{i,a}\left(\frac{1}{i}\frac{\delta}{\delta A_i^a(\vec{x})} - \frac{k}{8\pi}\varepsilon_{ij}A_j^a(\vec{x}) \right)^2 + \frac{1}{2e^2}\sum_a B^a(\vec{x})^2 \right) \Psi[A] \ = \ \mathcal{E} \ \Psi[A] \tag{20}$$

where $\mathcal{E}$ is the energy of the state. Among the stationary states, the physical states are those which are in the kernel of the gauge generator,

$$\left(D_i^{ab}(\vec{x}) \left(\frac{1}{i} \frac{\delta}{\delta A_i^b(\vec{x})} - \frac{k}{8\pi} \varepsilon_{ij} A_j^b(\vec{x}) \right) + \frac{k}{4\pi} B^a(x) \right) \Psi_{\text{phys}} [A] = 0 \qquad (21)$$

Since the gauge constraint commutes with the Hamiltonian, the gauge condition (21) can be consistently imposed on eigenstates of the Hamiltonian. The presence of the magnetic field term in $\mathcal{G}(\vec{x})$ implies that the gauge symmetry is represented projectively. The invariance of the physical states under infinitesimal gauge transformations in (21) can be integrated. Accordingly, the wavefunctional of a physical state obeys

$$\Psi_{\text{phys}} [A] = e^{i\alpha[A,g]} \Psi_{\text{phys}}[A^g] \qquad (22)$$

where the projective phase is given by the cocycle

$$\alpha[A, g] = \frac{-ik}{4\pi} \int d^2x \text{Tr}(A \, dgg^\dagger) - \frac{k}{4\pi} \Gamma[g] , \qquad (23)$$

related to the two-dimensional chiral anomaly (see. e.g. [18, 19]). The last term in this equation is the Wess-Zumino term [20], written in terms of the functional

$$\Gamma[g] = \frac{1}{3} \int_B \left(g^\dagger dg \right)^3 \qquad (24)$$

of the extension of the gauge transformation group element g to the 3-ball B, which has the space as its boundary. This term is well-defined when the group valued field, $g(\vec{x})$, has boundary conditions such that the space can be considered as one-point compactified. This is sufficient for the two dimensional space to be the boundary of a three-dimensional space. The Wess-Zumino term is well-defined up to an integer which does not appear in the transformation law for the wavefunctional if the coefficient k is quantized. This requirement of finding a single-valued solution to the gauge constraint is the reason for quantization of the coefficient of the Chern Simons term in the Hamiltonian formalism.

3 Strong coupling limit

The strong coupling limit, $e^2 \to \infty$, of the topologically massive gauge theory is obtained in the Hamiltonian formalism by neglecting the magnetic term in the Hamiltonian. Physically, this is the limit of large topological mass, where all momenta of interest are less than the topological mass $\mu = e^2k/4\pi$. In this limit, since the gluon has infinite mass, we expect that the model has no degrees of freedom.

Since the strong coupling theory is not generally renormalizable, it is usually considered in the context of lattice gauge theory [21]. In 2+1 dimensions, this is particularly obvious, since the coupling constant e^2 has dimensions of mass. Therefore, the large coupling limit, particularly of quantities with no momentum or coordinate dependence, can only be taken relative to some scale, such as the ultraviolet momentum cutoff, Λ^2. In a later section we shall make some conjectures

about how the coupling constant can be adjusted as a function of the cutoff so that some thermodynamic quantities turn out finite.

The strong coupling Hamiltonian is

$$H_0 = e^2 \int d^2x \; \mathrm{Tr} \left(\vec{E}^2(\vec{x}) \right) \tag{25}$$

The energy densities at different points commute,

$$\left[\; \mathrm{Tr} \left(\vec{E}^2(\vec{x}) \right) \; , \; \mathrm{Tr} \left(\vec{E}^2(\vec{y}) \right) \; \right] = 0 \tag{26}$$

and they also commute with $\mathcal{G}(\vec{x})$. Therefore, we can search for simultaneous eigenfunctionals of these *densities*.

Let us assume that k is positive. Consider the holomorphic factorization of the energy density

$$\mathrm{Tr} \left(\vec{E}^2(\vec{x}) \right) \;\; = \;\; \mathrm{Tr} \left(E^\dagger(\vec{x}) E(\vec{x}) \right) + \frac{k}{4\pi} \delta^2(\vec{0}) \tag{27}$$

where the "creation" and "annihilation" operators are

$$E(\vec{x}) = E_1(\vec{x}) - iE_2(\vec{x}) \;\; , \;\; E^\dagger(\vec{x}) = E_1(\vec{x}) + iE_2(\vec{x}) \;\; , \tag{28}$$

and the divergent term in Eq. (27) is the vacuum energy density arising from normal ordering the energy density using the commutator

$$\left[E^a(\vec{x}), E^{b\dagger}(\vec{y}) \right] = \frac{k}{2\pi} \delta^{ab} \delta(\vec{x} - \vec{y}) \tag{29}$$

Corresponding to the operators $E, E^\dagger$, we have the conjugate anti-holomorphic and holomorphic fields $A_\pm \equiv A_1 \pm iA_2$, with the non-vanishing commutators

$$\left[A_+^a(\vec{x}), \; E^b(\vec{y}) \right] = \left[A_-^a(\vec{x}), \; E^{b\dagger}(\vec{y}) \right] = 2i \; \delta^{ab} \delta(\vec{x} - \vec{y}) \tag{30}$$

As usual, we define the normal ordered energy density by dropping the divergent term:

$$: \mathrm{Tr} \left(E^2(\vec{x}) \right) : \;\; \equiv \;\; \mathrm{Tr} \left(E^\dagger(\vec{x}) E(\vec{x}) \right) \tag{31}$$

The vacuum state is the zero mode of the annihilation operator,

$$E^a(\vec{x}) \; \Psi_0[A] = \left(\frac{2}{i} \frac{\delta}{\delta A_+^a(\vec{x})} - i \frac{k}{8\pi} A_-^a(\vec{x}) \right) \Psi_0[A] = 0 \tag{32}$$

If normalizable solutions of this equation exist, then they can be superposed to form eigenstates of $\mathcal{G}(\vec{x})$. Those annihilated by $\mathcal{G}(\vec{x})$ are gauge invariant ground states.

The vacuum-equation is solved by any functional of the form

$$\Psi_0[A] = \psi[A_-] \; \exp \left\{ -\frac{k}{8\pi} \int d^2x \; \mathrm{Tr} \left(\vec{A}^2(\vec{x}) \right) \right\} \tag{33}$$

where ψ is a holomorphic wave-functional, depending only on A_- of the two components $A_\pm$. This solution is degenerate in that any functional ψ is acceptable,

subject to the normalizability of the wave-functional, implemented by functional integration

$$1 = \int [dA] \; \Psi_0^\dagger[A] \; \Psi_0[A] \tag{34}$$

which is consistent when $k > 0$. The degeneracy is resolved by requiring gauge invariance,

$$\mathcal{G}(\vec{x})\Psi_0[A] = 0 \tag{35}$$

This yields the equation

$$\left(D_-^{ab}(\vec{x}) \frac{\delta}{\delta A_-^b(\vec{x})} - \frac{k}{8\pi} \partial_+ A_-^a(\vec{x}) \right) \psi[A_-] = 0 \tag{36}$$

where $\partial_\pm = \partial/\partial x_1 \pm i\partial/\partial x_2$. This is the anomaly equation for k flavors of chiral fermions in two-dimensional Euclidean space[4]. A quantity that satisfies this equation is the k'th power of the determinant of the chiral Dirac operator in two dimensions

$$\psi[A_+] \;=\; \text{const.} \; [\det D_+]^k \tag{37}$$

where the chiral Dirac operators, in a Weyl basis, are

$$D_+ \;=\; \begin{pmatrix} 0 & \partial_- \\ -\partial_+ + iA_+ & 0 \end{pmatrix} \;,\quad D_- \;=\; \begin{pmatrix} 0 & \partial_- - iA_- \\ -\partial_+ & 0 \end{pmatrix} \tag{38}$$

The normalization integral for the wave-function is

$$Z = \int [dA] \; [\det D_-]^k \; [\det D_+]^k \exp \left\{ -\frac{k}{4\pi} \int d^2x \, \text{Tr}\left(\vec{A}^2(\vec{x}) \right) \right\} \tag{39}$$

Note that the terms in the exponent are exactly the counterterms which are necessary to convert the holomorphically factorized determinant of the Dirac operator into the gauge invariantly regulated determinant of the vector-coupled Dirac operator (see e.g. [24]),

$$Z \;=\; \int [dA] \; [\det \not{D}]^k \tag{40}$$

$$=\; \lim_{g^2 \to \infty} \int [dA][d\psi d\bar{\psi}] e^{-\int d^2x \left(\sum_{\alpha=1}^k \bar{\psi}_\alpha \, \not{D} \psi_\alpha + \frac{1}{2g^2}\text{Tr}F_{\mu\nu}^2 \right)} \tag{41}$$

where we have absorbed a factor of $\det^k(-\partial^2)$ into the normalization. In both (40) and (41), the Dirac determinant is assumed to have a gauge invariant regularization. Eq. (41) is the strong coupling limit of two dimensional QCD with k flavors of massless quarks. Furthermore, the computation of any observable functional of $\vec{A}(\vec{x})$ is equivalent to the computation of the equivalent expectation value in the two-dimensional Euclidean gauge theory

$$\langle 0|\mathcal{O}[A]|0\rangle = \lim_{g^2 \to \infty} \frac{1}{Z} \int [dA][d\psi d\bar{\psi}] \; \mathcal{O}[A] e^{-\int d^2x \left(\sum_{\alpha=1}^k \bar{\psi}_\alpha \, \not{D} \psi_\alpha + \frac{1}{2g^2}\text{Tr}F_{\mu\nu}^2 \right)} \tag{42}$$

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[4]The holomorphicity (33) and gauge-invariance (36) conditions on the vacuum wave-functional are familiar from the Bargmann quantization of Chern-Simons theory [3, 22, 23], where the space of spatial gauge fields on a Riemann surface, subjected to a gauge constraint generated by a flatness condition, is quantized with respect to the Kähler potential $\int A_+ A_-$. The appearance of these conditions just rephrases the fact that Chern-Simons theory describes the vacuum sector of strongly coupled gauge theory.

For example, the vacuum expectation value of the spatial Wilson loop in 2+1-dimensional gauge theory is the expectation value of a spacetime Wilson loop in the infinite coupling limit of two-dimensional QCD. In the latter theory, since the quarks transform under the fundamental representation of the gauge group, the Wilson loop exhibits a perimeter law,

$$\left\langle \text{Tr} \left(\mathcal{P} e^{i \oint_\Gamma A} \right) \right\rangle \sim \exp \left\{ -\sigma L[\Gamma] \right\} \tag{43}$$

This is a signal of the magnetic mass of the topological field theory and is consistent with the fact that topologically massive gauge theory is not confining.

4 Bosonization, Polyakov loop correlators and the Verlinde formula

To facilitate computations, we bosonize the functional integral (42) using non-Abelian bosonization [16]. Two-dimensional fermion determinants in a background field A were computed by Polyakov and Wiegmann [25], resulting in a principal chiral model with a Wess-Zumino term. The corresponding functional is the celebrated Wess-Zumino-Novikov-Witten action [15, 16],

$$I[h] = \frac{1}{8\pi} \int d^2x \, \text{Tr} \, |\nabla h|^2 + \frac{i}{4\pi} \Gamma[h] \, , \tag{44}$$

with Γ the Wess-Zumino functional of Eq. (24). The group valued field h is an hermitian combination $h = \tilde{h}\tilde{h}^\dagger$ of a field $\tilde{h}$ taking values in the complexification $\text{G}_\mathbb{C}$ of the gauge group, which describes the background field as

$$A_+ = i\tilde{h}^{-1}\partial_+\tilde{h} \, , \quad A_- = i\partial_-\tilde{h}^\dagger(\tilde{h}^\dagger)^{-1} \, . \tag{45}$$

Using the Polyakov-Wiegmann formula,

$$I[\tilde{h}\tilde{h}^\dagger] = I[\tilde{h}] + I[\tilde{h}^\dagger] - \frac{1}{4\pi} \int d^2x \, \tilde{h}^{-1}\partial_+\tilde{h} \, \partial_-\tilde{h}^\dagger(\tilde{h}^\dagger)^{-1} \, , \tag{46}$$

we get the fermion determinant exactly in the holomorphically factorized form corresponding to Eq. (39).

The WZNW functional describing the fermion determinant is invariant under vector gauge transformations, $\tilde{h} \to \tilde{h}g$, $\tilde{h}^\dagger \to g^\dagger\tilde{h}^\dagger$. However, under a chiral gauge transformation, $\tilde{h} \to \tilde{h}g$, $\tilde{h}^\dagger \to g\tilde{h}^\dagger$, it transforms non-trivially. Using this property, we can write an equivalent bosonic action for the fermionic functional integral, in terms of the group valued field $g \in \text{G}$ [26]. The resulting theory is a gauged WZNW theory [17], more exactly the so called G/G WZNW theory with the action

$$S_{\text{G/G}}[g, A] = I[g] + \frac{1}{4\pi} \int d^2x \text{Tr} \left(iA_-\partial_+gg^{-1} + A_+A_- - A^g_-A_+ \right) \, . \tag{47}$$

Here we have chosen to use the gauge fields instead of the group valued field $\tilde{h}$.

For a generic topological mass, we are dealing with the k:th power of the fermion determinant, so the required bosonization is somewhat more complicated [27]. In

addition to the gauged WZNW model for the color degrees of freedom, now at level k, there are ungauged ones for the flavor degrees of freedom and for a field parametrizing the coset of the total ungauged symmetry group of the fermions divided by the gauge and flavor subgroups of the total symmetry group. Here we shall only be interested in the color degrees of freedom, which decouple from the rest as there is no mass term for the would-be bosonized fermions. The normalization integral for the vacuum wave functional (41) thus becomes

$$Z = \int [dA][dg] \mathrm{e}^{-k S_{\mathrm{G/G}}} \ , \tag{48}$$

which is the partition function for the G/G model at level k.

The result that the vacuum sector of strongly coupled topologically massive gauge theory is described by a G/G WZNW model is by no means surprising. The connections of pure Chern-Simons theory, which describes the vacuum sector, with G/G models have been widely investigated in the literature [8, 9, 10, 11]. In particular, in Refs. [10, 11] it was shown that the finite temperature effective action for Chern-Simons theory, defined on the three-manifold $\Sigma \otimes S^1$ with Σ an arbitrary Riemann surface, is nothing but the G/G action. There is no temperature dependence, as the Hamiltonian of pure Chern-Simons theory is zero.

Moreover, the properly normalized partition function of G/G WZNW on a Riemann surface Σ is an integer, characterizing the dimension of the Chern-Simons Hilbert space when quantized on $\Sigma \otimes \mathbb{R}$. These integers are given by the so-called Verlinde formula [28], where they appeared as the dimensions of the spaces of conformal blocks of a WZNW model for the group G. We shall be interested in the simplest case when Σ is a sphere, possibly with punctures.

The G/G partition function on a sphere is just 1, confirming the uniqueness of our vacuum (33). Punctures, on the other hand, are labeled by representations λ_i of the group. Most interesting for us is the observation [3, 10] that punctures on Σ correspond to traces of vertical Wilson loops in representations λ_i when quantizing Chern-Simons theory on $\Sigma \otimes S^1$, i.e. nothing but the Polyakov loops of finite temperature gauge theory [29, 30].

Polyakov loops in Chern-Simons theory correspond in G/G to traces of the group-valued field g, i.e. characters $\mathcal{X}_{\lambda_i}(x_i) \equiv \mathrm{Tr}[g(x_i)]_{\lambda_i}$ in the representations λ_i.

The correlation functions

$$\left\langle \prod_i \mathcal{X}_{\lambda_i}(x_i) \right\rangle = \int [dA][dg] \prod_i \mathcal{X}_{\lambda_i}(x_i) \, \mathrm{e}^{-k S_{\mathrm{G/G}}[g,A]} \tag{49}$$

are given by the Verlinde formula for the dimensions of the space of conformal blocks on the corresponding punctured surfaces. Specializing to the case of gauge group G=SU(2), and integrable ($\lambda \leq k$) representations of the corresponding Kač-Moody group, we have the lowest-point functions (see e.g. [10])

$$\langle \mathcal{X}_\lambda(x) \rangle = 0 \ ; \quad \lambda > 0 \tag{50}$$

$$\langle \mathcal{X}_\lambda(x) \mathcal{X}_\nu(y) \rangle = \delta_{\lambda\nu} \tag{51}$$

$$\langle \mathcal{X}_\lambda(x)\mathcal{X}_\mu(y)\mathcal{X}_\nu(z)\rangle = N_{\lambda\mu\nu} \qquad (52)$$

Here $N_{\lambda\mu\nu}$ is the matrix encoding the fusion rules of conformal blocks corresponding to different primary fields inserted at the punctures.

As explained above, this enables us to calculate correlators of Polyakov loops in any representation. As G/G WZNW theory is a topological field theory, the correlators are just topological invariants pertaining to the corresponding punctured spheres. These correlators can then be interpreted as zero temperature correlators in strong coupling topologically massive gauge theory. For pure Chern-Simons theory, corresponding to infinite coupling, $e^2 = \infty$, expressions (50)-(52) give the Polyakov loop correlators in the full finite temperature theory. The latter is of course rather trivial, as in a topological theory there are no states at finite energy and therefore no excited states to give a temperature dependence to the partition function or correlators.

Polyakov loops correspond to infinitely massive external particle insertions, and their correlators are of vital importance when investigating the confinement properties of gauge theories in the absence of dynamical fundamental fermions [29, 30]. This is due to the fundamental representation Polyakov loop's ability to probe the spontaneous breaking of the hidden symmetry of the theory under global transformations in the center of the gauge group (see Ref. [31] for a comprehensive review).

Here one has to notice that the results for Polyakov loop correlators quoted above are computed on a finite volume Riemann surface. Eq. (50) indicates that the expectation value of a Polyakov loop is always vanishing in finite volume. On the other hand, cluster decomposition of the correlator (51) with fundamental representation Polyakov and anti-Polyakov loops gives a *non-vanishing* expectation value for the Polyakov loop in the infinite volume limit. This indicates spontaneous breaking of the center symmetry in infinite volume. According to [29, 30], this symmetry breaking indicates the de-confinement of color in the theory. This is commensurate with what we would expect for this model, since the gluons are very massive, and therefore incapable of mediating a confining interaction.

For topologically massive gauge theory, eqs. (50)-(52) give vacuum correlators of Polyakov loops. To investigate the full thermodynamical expectation values, one has to find the effective thermodynamical theory for $e^2 \neq \infty$. This will be done in the next section. However, there is no reason to expect that the long distance correlations expressed in (51) would vanish by exiting some local quantum states. Accordingly, we expect the center symmetry to be broken in the topologically massive theory as well, which is consistent with the fact that the latter theory is not confining.

The screening and confinement properties of the Abelian topologically massive theory (at finite temperature and any coupling) were investigated in Ref. [32]. There it was noticed that the effective action is of the same Villain form as the one obtained for the Schwinger model in [33], although one dimension higher. Detailed calculations in [33] show an analogous picture as the one emerging here from the analysis of Eqs. (50) and (51) in the $e^2 = \infty$ case: At finite volume, the Polyakov

loop expectation value is always zero. In the infinite volume limit one has to rely on cluster decomposition of the correlator of two Polyakov loops. The center symmetry is broken and the theory is in a deconfined phase.[5]

5 Finite temperature theory

Let us now consider the strongly coupled theory at finite temperature. The partition function is the trace of the Boltzman factor over the physical states. We will use the density matrix

$$\rho = e^{-H/T} \cdot \mathcal{P} \tag{53}$$

where H is the Hamiltonian and $\mathcal{P}$ is the projection operator onto physical states. To choose physical states, we shall use position eigenstates which have the property

$$\hat{\vec{A}}(\vec{x}) \, |A\rangle = \vec{A}(\vec{x}) \, |A\rangle \tag{54}$$

We project onto gauge invariant states by integrating over all gauge copies of the state

$$\mathcal{P} \, |A\rangle = \frac{1}{\text{vol } \mathcal{G}} \int [dg] \, e^{i\alpha[A,g]} \, |A^g\rangle \tag{55}$$

where $[dg]$ is a local Haar measure and vol $\mathcal{G}$ is the volume of the gauge group. Using this, the partition function is (up to an overall constant)

$$Z[T] = \int [dg] dA \, e^{i\alpha[A,g]} \langle A| \, e^{-H/T} \, |A^g\rangle \tag{56}$$

We shall consider the strong coupling limit where $H \to H_0$. This limit could be considered as the limit $e^2 T \to \infty$ (again since this is a dimensionless number, it should be read as infinity compared to all other mass or momentum scales of interest) while keeping the dimensionless ration e^2/T constant. We will compute the propagator

$$\langle A| \, e^{-H_0/T} \, |A^g\rangle \, .$$

The propagator satisfies the Heat equation,

$$\left[\frac{\partial}{\partial T^{-1}} + H_0 \right] \langle A| \, e^{-H_0/T} \, |A'\rangle = 0 \, , \tag{57}$$

with the boundary condition

$$\lim_{T \to \infty} \langle A| \, e^{-H_0/T} \, |A'\rangle = \prod_{x,a,i} \delta(A_i^a(\vec{x}) - A_i^a(\vec{x})') \, . \tag{58}$$

This equation is solved by the Gaussian functional

$$\langle A| e^{-H_0/T} |A'\rangle = \frac{\exp\left\{ -\frac{k}{8\pi} \int d^2x \ \text{Tr} \left(\coth\left(\frac{ke^2}{8\pi T} \right) \left(\vec{A} - \vec{A}' \right)^2 + 2i\vec{A} \times \vec{A}' \right) \right\}}{\left(\frac{16\pi^2}{k} \sinh\left(\frac{e^2 k}{8\pi T} \right) \right)^{(\dim \ G) \ V \ \delta(0)}} \tag{59}$$

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[5]In the Abelian theories, the analogue of the center symmetry is the Z symmetry of the finite temperature theory, and the order parameter transforming under this symmetry, corresponding to an fundamental Polyakov loop, is the Polyakov loop of a particle with an incommensurate charge compared to the basic charge of the theory.

where V is the volume of the two-manifold Σ.

Combining the propagator with the cocycle $e^{i\alpha[A,g]}$, the result for the partition function is

$$Z[T] = 1/\mathcal{N} \int [dg(\vec{x})][d\vec{A}(\vec{x})] \; e^{-S_{\text{eff}}[A,g]} \; , \tag{60}$$

where the finite temperature effective action is [6]

$$S_{\text{eff}}[A, g] \;\; = \;\; \frac{k}{8\pi} \coth\left(\frac{ke^2}{8\pi T}\right) \int d^2x \; \text{Tr}|\vec{D}g|^2 + \frac{ik}{4\pi}\Gamma[g, A] \; ; \tag{61}$$

$$\Gamma[g, A] \;\; = \;\; \Gamma[g] + \text{Tr} \int \vec{A} \times \left(\vec{A}^g + i\vec{\partial}gg^\dagger\right) \; , \tag{62}$$

and the normalization is given by

$$\mathcal{N} = \left(\frac{16\pi^2}{k} \sinh\left(\frac{e^2 k}{8\pi T}\right)\right)^{(\dim \; G)) \; V \; \delta(0)} . \tag{63}$$

This is the gauged principal chiral model with a gauged Wess-Zumino term, i.e. a perturbed G/G WZNW model. In the limit where $\coth(e^2k/8\pi T) \to 1$, which is achieved by $ke^2/T \to \infty$, it approaches the level k G/G-model. The effective action (61-62) can also be obtained starting from the propagator of a particle in a constant magnetic field [34] and extending the result to infinite degrees of freedom. At each point the hamiltonian density in the strong coupling limit, is in fact the hamiltonian of a particle of mass $1/e^2$ in a magnetic field $k/4\pi$.

Perturbations of G/G WZNW models of the form (61) have been earlier discussed by Witten [12] and Blau & Thompson [13]. In [12] it was argued that the theory is topological at the classical level, but the topological invariance is broken due to quantum effects. In [13], a supersymmetrized version of (61) was investigated. The supersymmetrization was chosen so that the perturbed model was equivalent to the unperturbed G/G model in the equivariant localization sense. Here, we shall consider the truly perturbed G/G model, and find that the topological invariance is indeed broken.

6 One loop effective potential for g

Now we shall investigate the finite temperature effective field theory built on the perturbed G/G action (61). We shall mainly be interested in the effects of the perturbation $k \to k \coth \frac{ke^2}{8\pi T}$ [7].

By integrating out the quantum fluctuations to one-loop order we shall find the effective potential for the group valued field g.

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[6]The covariant derivative acts on g with the adjoint action $D_\mu g(x) = \partial_\mu g(x) - i\left[A_\mu(x), g(x)\right]$, so that $g^\dagger D_\mu g$ equals the finite gauge transformation $A_\mu^g - A_\mu$.

[7]It is a well known fact, both in G/G models [10, 11, 13] and more generally in topologically massive gauge theory [35, 36] that the level k gets shifted by quantum effects to $k + c_v$, where c_v is the quadratic Casimir in the adjoint representation. In this paper we are not interested in this issue.

We expand g in terms of the constant background field g_o and algebra valued fluctuations $\sigma(x)$[8];

$$g = g_o \, \mathrm{e}^{i\sigma} \ .$$

The piece of the finite temperature effective action (61) quadratic in the quantum fields is

$$S_q[g_o; \sigma, A_i] = \frac{1}{4\pi} \int \mathrm{Tr}\Big\{ \frac{\kappa}{2}(\partial_i \sigma)^2 \ - \ 2ik\varepsilon^{ij} A_i \partial_j \sigma \\ + \ (\mathbb{1} - \mathrm{Ad}_{g_o})\,(A_i)\mathcal{P}^{ij}(A_j + \partial_j\sigma) \Big\} \ , \tag{64}$$

where Ad is the adjoint action of the group, $\mathcal{P}^{ij} = \kappa\delta^{ij} + ik\varepsilon^{ij}$ is the deformed projector onto holomorphic forms, and

$$\kappa = k \coth\left(\frac{ke^2}{8\pi T} \right) \tag{65}$$

is the perturbed level.

In order to regulate the integration over the gauge field A, we add the Yang-Mills term multiplied by an infinitesimal parameter, $\epsilon \int \mathrm{Tr}(F^2)/8\pi$, to the action. In the original (2+1)-dimensional language, this regulator is the magnetic field squared term which was killed by the strong coupling limit. We shall later take the limit where its coefficient ϵ goes to zero like $\epsilon = 1/\Lambda^2$. We shall also add a covariant gauge fixing term $\lambda \int \mathrm{Tr}(\nabla \cdot A)^2/4\pi$.

Defining

$$R^{ab}(g) = 2\,\mathrm{Tr}\left((\mathbb{1} - \mathrm{Ad}_g)(T^a)\,T^b \right) = \delta^{ab} - 2\,\mathrm{Tr}\left(gT^a g^{-1}T^b \right) \ , \tag{66}$$

we can write the action in terms of the coordinate fields in the Lie-Algebra:

$$S_q = \frac{1}{8\pi} \int \Big\{ -\frac{\kappa}{2}\sigma^a \Delta\sigma^a \ - \ \left(R^{ab}(g_o)\mathcal{P}^{ij} - 2ik\delta^{ab}\varepsilon^{ij} \right) \sigma^b \partial_j A_i^a \\ + \ A_i^a R^{ab}(g_o)\mathcal{P}^{ij} A_j^b \ + \ \epsilon(\partial_i A_j^a - \partial_j A_i^a)\partial_i A_j^a \ + \ \lambda \partial_i A_i^a \, \partial_j A_j^a \Big\} \ . \tag{67}$$

The σ field is now easily integrated out. In order to facilitate the ensuing A-integration, we change integration variables to the curl and divergence of A. Using the identity

$$\delta^{ij}(x - y) \ = \ \varepsilon^{ik}\partial_k \frac{1}{\Delta}\varepsilon^{jl}\partial_l \ + \ \partial_i \frac{1}{\Delta}\partial_j$$

we can write the term quadratic in A in terms of $C^a = \nabla \times A^a$ and $D^a = \nabla \cdot A^a$. The Jacobian arising from the change $A_i^a \to \{C^a, D^a\}$ is just proportional to the determinant of the Laplacian. After integrating over σ^a, we have the effective action

$$S_{\tilde{q}} = \frac{1}{8\pi} \int \Big\{ \frac{1}{\kappa}\left(2k^2\delta^{ab} + (\kappa^2 - k^2)R^{(ab)} \right) C^a \frac{-1}{\Delta} C^b \ + \ \epsilon(C^a)^2 \ + \ \lambda(D^a)^2 \Big\} \ . \tag{68}$$

In deriving (68) we used properties of R that follow directly from its relation (66) to the adjoint representation matrix. The symmetrized matrix $R^{(ab)}$ is Hermitean, so we can diagonalize it. We denote the corresponding Eigenvalues r^a.

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[8]Note that with a constant background the expansion in terms of the coordinates σ, coincides with the normal coordinate expansion.

Next, we integrate out D^a and C^a. This gives the one-loop contribution to the partition function

$$Z_q \sim \prod_a \left[\text{Det} \left(\frac{2k^2 - \epsilon\kappa\Delta}{-\kappa\Delta} \right) \text{Det} \left(1 + \frac{(\kappa^2 - k^2)\ r^a(g_o)}{2k^2 - \epsilon\kappa\Delta} \right) \right]^{-1/2} .$$

From this, we can read out the effective potential for g_o:

$$V_{\text{eff}}[g_o] = \frac{V}{2} \int \frac{d^2p}{4\pi^2}\ \ln \left[1\ +\ \frac{(\kappa^2 - k^2)\ \text{Tr}\ R(g_o)}{2k^2 + \epsilon\kappa p^2} + \mathcal{O}\left((\kappa^2 - k^2)^2 \right) \right] . \qquad (69)$$

The matrix R is traced over adjoint indices, $Tr\ R = N^2 - |\text{Tr}\ g|^2$. The expansion is around the zero temperature (infinite coupling) conformal fixed point with $\kappa = k$, which corresponds to the G/G-model.

With a fundamental cut-off Λ, we can perform the integration over p. In the vicinity of the conformal fixed point we thus have

$$V_{\text{eff}} \sim \Lambda^2 \ln \left(1\ +\ \frac{a}{b + \epsilon\Lambda^2} \right)\ +\ \frac{a+b}{\epsilon} \ln \left(1\ +\ \frac{\epsilon\Lambda^2}{a+b} \right)\ -\ \frac{b}{\epsilon} \ln \left(\frac{b + \epsilon\Lambda^2}{b} \right) . \qquad (70)$$

Here we have denoted $a = \text{Tr}\ R(g_o)\ (\kappa^2 - k^2)/\kappa$ and $b = 2k^2/\kappa$. This indicates that the effective potential is finite if the regulator is taken to zero according to $\epsilon \sim \Lambda^{-2}$. In addition, we have to require that $a \sim Tr\ Rm^2/\Lambda^2$, for some scale μ. Recalling the definition (65) of κ, this means that e^2/T has to scale logarithmically with the cutoff.

Our effective theory (61) for the thermodynamics of strongly coupled topologically massive gauge theory thus makes sense as long as we stay within a logarithmic distance from the cutoff, i.e. within a coupling-to-temperature range $e^2/T >\sim \ln\Lambda/\mu$.

The effective potential (70) has absolute minima for all the gauge group elements belonging to the center of the gauge group, $g_0 \in Z_N$. For such a g_0, $\text{Tr}\ R(g_0) = a = 0$, and the one-loop effective potential vanishes. Consequently, the Polyakov loop operators tend to be elements of Z_N so that the center symmetry is broken. This leads to a non-vanishing vacuum expectation value for a Polyakov loop operator, consistently with the discussion of Section 4 on the correlators of Polyakov loop correlators. We can then conclude that in the strong coupling limit topologically massive gauge theory is always in a deconfined phase, as one would expect in this regime where the gluons have a big mass.

7   Excited states

In this Section, we shall find the excited states of the theory, which give rise to the finite temperature partition function (60) in the regime for the coupling constants, discussed in the previous Section.

The natural candidate for excited states are the Landau levels created by the field theoretic harmonic oscillator creation operator $E^\dagger$ of Eq. (28), acting on the vacuum (33). The electric field transforms covariantly under a gauge transformation,

so some care has to be taken to render the emerging states gauge invariant up to the projective phase of Eq. (22). We shall see that this will be possible by employing the Chern-Simons Hilbert spaces on surfaces with punctures which were discussed in Section 4.

Using the results of Ref. [25] cited in Eqs. (44—46), we would straight forwardly get the holomorphically factorized ground state. To work within this formulation, we would then have to rewrite the electric fields (the gauge field functional derivatives) in terms of cotangent operators on the complex group manifold parametrized by $\tilde{h}$. This approach was taken in Refs. [37], where the related problem of the origin of the mass gap in 2+1 dimensional gauge theories was addressed.

For our purposes, it will be most transparent to avoid the inherent nonlinearities associated with quantizing the curved group manifold, and work on the level of the Lie-algebra instead. This can be achieved in the gauged WZNW formulation. The price one has to pay for working with a vector space formulation of the gauge fields, is increased complications in the holomorphic factorization. The inclusion of the bosonic field g spoils the straight forward factorizability given by the Polyakov-Wiegmann formula (46).

In Ref. [9] Witten showed that gauged WZNW models can be factorized only if one introduces an extra gauge field B (see also [38]). By introducing this field, and using the Polyakov-Wiegmann formula, as well as the invariance of the Haar measure, it is easy to show that the gauged WZNW partition function can be factorized as

$$Z = \int [dA][dg] \mathrm{e}^{-kS_{\mathrm{G/G}}} = \tfrac{1}{\mathrm{vol}\,\mathcal{G}} \int [dA][dB] \; \Psi_0^*[A, B] \; \Psi_0[A, B] \;, \qquad (71)$$

where the vacuum wave functional depending on the two gauge fields is

$$\Psi_0[A, B] = \mathrm{e}^{-\frac{k}{8\pi} \int \left(|A|^2 + |B|^2 \right)} \int [dg] \mathrm{e}^{-I[g] + \frac{k}{4\pi} \int \left\{ A_- B_+^g - iB_+ \partial g g^{-1} \right\}} \;. \qquad (72)$$

The extra gauge field B records the fact that the group G in the gauged WZNW model is diagonally gauged. This translates into the symmetry of $\Psi_0[A, B]$ under an exchange of A and B and complex conjugation:

$$\Psi_0^*[A, B] = \Psi_0[B, A] \;. \qquad (73)$$

Under a gauge transformation, $\Psi_0[A, B]$ transforms as

$$\Psi_0[A^h, B^l] = \mathrm{e}^{-i\alpha[A, h] \,+\, i\alpha[B, l]} \; \Psi_0[A, B] \;, \qquad (74)$$

generalizing the projective transformation of physical states (22) to two gauge fields. Similarly, it is easy to see that $\Psi_0[A, B]$ is annihilated by the annihilator E:

$$E^a \; \Psi_0[A, B] = \left(\frac{2}{i} \frac{\delta}{\delta A_+^a} - i \frac{k}{8\pi} A_-^a \right) \Psi_0[A, B] = 0 \;. \qquad (75)$$

The wave functional $\Psi_0[A, B]$ thus fulfills the requirements for a vacuum wave functional for the strongly coupled massive gauge theory. Moreover, due to Symmetry (73) the analogous operator in B acts as an annihilator as well:

$$\left(\frac{2}{i} \frac{\delta}{\delta B_-^a} - i \frac{k}{8\pi} B_+^a \right) \Psi_0[A, B] = 0 \;. \qquad (76)$$

In order to build gauge-invariant excited states, we take a look at Chern-Simons Hilbert spaces on punctured Riemann spheres. As explained in Section 4, punctures x_i correspond to Polyakov loop insertions, which again correspond to non-dynamical external particles. The particles, and accordingly the punctures, are characterized by the representations λ_i of the group G, under which they transform. The corresponding wave functionals can be acquired by inserting the group element g in representations λ_i at the points x_i, into the vacuum functional $\Psi_0[A, B]$ without punctures:

$$\Xi_{\{\lambda_i\}}[\{x_i\}; A, B] = \mathrm{e}^{-\frac{k}{8\pi} \int \left(|A|^2+|B|^2\right)} \int [dg] \otimes_i g_{\lambda_i}(x_i) \mathrm{e}^{-I[g]+\frac{k}{4\pi} \int \left\{A_- B_+^g -iB_+ \partial g g^{-1}\right\}} \quad (77)$$

The representations λ_i act on vector spaces V_i, so the wave functional is now an operator on the tensor product of the corresponding representation spaces,

$$\Xi_{\{\lambda_i\}}[\{x_i\}; A, B] \in \otimes_i \ (V_i^* \otimes V_i) . \quad (78)$$

Under gauge transformations, these functionals transform as

$$\Xi_{\{\lambda_i\}}[\{x_i\}; A^h, B^l] = \mathrm{e}^{i\alpha[B,l]-i\alpha[A,h]} \otimes_i \left(l^{-1}\right)_{\lambda_i}(x_i) \ \Xi_{\{\lambda_i\}}[\{x_i\}; A, B] \otimes_i h_{\lambda_i}(x_i) . \quad (79)$$

Moreover, the property of being annihilated by E is clearly insensitive to insertions of g, so these states are indeed vacuum states for our strongly coupled gauge theory in the presence of external particles.

With the same methods that were used to prove (71), it is easily seen that the normalization integral for this wave functional reproduces the G/G partition function with character insertions,

$$\frac{1}{\mathrm{vol}\,\mathcal{G}} \ \mathrm{Tr} \int [dA][dB] \ \Xi_{\{\lambda_i\}}^*[\{x_i\}; A, B] \ \Xi_{\{\lambda_i\}}[\{x_i\}; A, B]$$
$$= \int [dA][dg] \ \prod_i \mathcal{X}_{\lambda_i}(x_i) \ \mathrm{e}^{-k \ S_{\mathrm{G/G}}[g,A]} . \quad (80)$$

The trace is over all the representation spaces $V_i^* \otimes V_i$. The Verlinde formula then tells us, which combinations of insertions give wave functionals $\Xi_{\{\lambda_i\}}[\{x_i\}; A, B]$ with non-zero norms, and which are the true dimensions of the Hilbert spaces spanned by the states Ξ.

Here we are interested in insertions in the conjugate adjoint representation. These transform conjugate covariantly, which counteracts the covariant transformation of the electric field "creation operator" $E^\dagger$. For concreteness, we consider insertions of matrices
$$\left(g_{\mathrm{Ad}}^T\right)^{ab} = 2 \ \mathrm{Tr} \left(g T^b g^{-1} T^a\right) \quad (81)$$
at the points x_i, where the trace, the generators and the group element g are in the fundamental representation. Denoting a wave functional with m such insertions by $\Xi_m^{\{b_i,a_i\}}[\{x_i\}; A, B]$, we can act with the creation operators $E^{b\dagger}$ at the insertion points to acquire an excited state that is gauge invariant up to a projective phase:

$$\Psi_m^{\{a_i\}}[\{x_i\}; A, B] = \left(\frac{4\pi i}{k}\right)^m \prod_{i=1}^m E^{b_i\dagger}(x_i) \ \Xi_m^{\{b_i,a_i\}}[\{x_i\}; A, B]$$

$$= e^{-\frac{k}{8\pi} \int \left(|A|^2 + |B|^2\right)} \int [dg] \prod_{i=1}^{m} \left(B_+ - A_+^{g^{-1}}\right)^{a_i} (x_i) \ e^{-I[g] + \frac{k}{4\pi} \int \left\{A_- B_+^g - iB_+ \partial g g^{-1}\right\}} . \quad (82)$$

These states are tensor products of vectors in the adjoint representation space V_{Ad}, or equivalently, of fundamental representation matrices, $\Psi_m = \Psi_m^{\{a_i\}} \otimes_{i=1}^{m} T^{a_i} \in \otimes_{i=1}^{m} V_{\mathrm{Ad}}$. By construction, they are eigenstates of the Hamiltonian $H = \frac{e^2}{2} \int E^\dagger(\vec{x}) E(\vec{x})$. The energies of these excited states are those of Landau levels,

$$H \ \Psi_m[\{x_i\}; A, B] \ = \ m \frac{e^2 k}{4\pi} \ \Psi_m[\{x_i\}; A, B] , \quad (83)$$

and there is a continuous degeneracy labeled by the locations of the insertions, $\{x_i\}$. These Landau levels thus consist of m gauge invariant combinations of external particles and glue, situated at the points x_i.

The connection of these exited states to the ones found in Refs. [37] is not entirely straightforward. There, WZNW theory was used to describe the Jacobian arising from the change of variables (45), for the strong coupling theory with $k = 0$. It is well known [25] that this Jacobian gives rise to a WZNW theory with level c_v, related to the renormalization of k. Within this framework, the authors of [37] constructed exited states depending of the gauge currents. These states should be related to the ones found here, upon eliminating the extra gauge field B. In our formalism, the shift in the energy level can be understood as arising from the short distance singularity in E^2, see e.g. [39].

The inner product of two states of the form (82) can be constructed directly by tracing over the representation spaces V_{Ad} and integrating over the gauge fields A and B. Summing over permutations P of the positions x_i of the m external particles, we have

$$(\Psi_m[\{x_i\}], \Psi_n[\{y_i\}]) = \delta^{mn} \sum_{P_m} \int [dA][dB][dg][dh] \ e^{-I[g] - I[h]}$$

$$\times \ \exp\left\{-\frac{k}{4\pi} \int \left[|A|^2 + |B|^2 - B_- A_+^g - B_+ A_-^{h^{-1}} + iA_+ \partial g g^{-1} - iA_- h^{-1} \bar\partial h\right]\right\}$$

$$\times \ \prod_{i=1}^{m} \mathrm{Tr} \left[\left(B_- - A_-^g\right)(x_i) \left(B_+ - A_+^{h^{-1}}\right)(y_{P(i)})\right] \quad (84)$$

where the inner product of the vectors in V_{Ad} is normalized to coincide with the fundamental representation trace. This Gaussian path integral is easily performed. The most transparent way is to introduce the change of variables

$$B_+ \to hB_+ h^{-1} + A_+^g \ , \quad B_- \to hB_- h^{-1} + A_+^{h^{-1}} \quad (85)$$

with unit Jacobian. After performing this, the inner product can be expressed in terms of Gauged WZNW actions $S_{G/G}[g, A]$ and gauge currents $J(g) = A^g - A$,

$$(\Psi_m[\{x_i\}], \Psi_n[\{y_i\}])$$

$$= \delta_{mn} \times \sum_{P_m} \int [dA][dB][dg][dh] \prod_{i=1}^{m} \mathrm{Tr} \left[\left(B_- - J_-(gh) \right)(x_i) \left(B_+ + J_+(gh) \right)(y_{P(i)}) \right]$$

$$\times \ \exp \left\{ -S_{G/G}[g, A] - S_{G/G}[h, A] + \tfrac{k}{4\pi} \int \left(J_+(g) \ J_-(h^{-1}) \ - \ |B|^2 \right) \right\} (86)$$

Now we can use the gauge-generalized Polyakov-Wiegmann formula

$$S_{G/G}[gh, A] = S_{G/G}[g, A] + S_{G/G}[h, A] - \tfrac{k}{4\pi} \int J_+(g) \ J_-(h^{-1}) \tag{87}$$

and the invariance of the Haar measure to transform away the h integration. The inner product reduces to the unperturbed G/G expectation value

$$(\Psi_m[\{x_i\}], \Psi_n[\{y_i\}]) = \mathrm{vol} \ \mathcal{G} \ \ \delta_{mn} \left\langle \mathcal{Z}_m \left(\{x_i\}, \{y_i\} \right) \right\rangle_{G/G} \ , \tag{88}$$

where

$$\mathcal{Z}_m \left(\{x_i\}, \{y_i\} \right) = \sum_{P_m} \int [dB] \ \mathrm{e}^{-\frac{k}{4\pi} \int |B|^2} \ \prod_{i=1}^{m} \mathrm{Tr} \left[\left(B_- - J_- \right)(x_i) \left(B_+ + J_+ \right)(y_{P(i)}) \right] \tag{89}$$

$$= \sum_{P_m} \sum_{l=0}^{m} \left(\tfrac{4\pi}{k} \right)^l \sum_{\{x_s\}_{s=1}^l \subset \{x_i\}_{i=1}^m} (\dim G)^l \prod_s \delta(x_s - y_P(s)) \prod_{r \neq s} \mathrm{Tr} \left[-J_-(x_r) J_+(y_{P(r)}) \right]$$

The expectation values of G/G currents can be reduced to expectation values of the operators R of Eq. (66),

$$\left\langle \prod_{i=1}^{m} J_-^{a_i}(x_i) \ J_+^{b_i}(y_i) \right\rangle_{G/G} = \sum_{P} \left(\tfrac{8\pi}{k} \right)^m \left\langle \prod_{i=1}^{m} R^{a_i b_{P(i)}}(x_i) \right\rangle_{G/G} \delta(x_i - y_{P(i)}) \ , \tag{90}$$

Remembering that $\sum_a R^{aa} = \dim G \ - \mathcal{X}_{\mathrm{Ad}}$, we get for the inner product

$$(\Psi_m[\{x_i\}], \Psi_n[\{y_i\}]) = \mathrm{vol} \ \mathcal{G} \ \delta_{mn} \left(\tfrac{4\pi}{k} \right)^m \left\langle \prod_{i=1}^{m} \mathcal{X}_{\mathrm{Ad}}(x_i) \right\rangle_{G/G} \sum_{P_m} \prod_{i=1}^{m} \delta(x_i - y_{P(i)}) \ . \tag{91}$$

The states $\Psi_m[\{x_i\}]$ are thus orthogonal, and delta-function normalizable with the norm $\left(\tfrac{4\pi}{k} \right)^{m/2} \sqrt{\mathrm{vol} \ \mathcal{G}}$. The G/G expectation value of the characters again expresses the dimensionality of the Hilbert space spanned by the components of the wave functional Ψ_m, which is the same as the dimensionality of the space spanned by the underlying states Ξ_m. These are the topological invariants given by the Verlinde formula.

As the orthogonal excited states Ψ_m fulfill the projective gauge transformation property (22), their contribution to the finite temperature partition function is straight forward to compute using the results above. To connect with the path integral formulation of Section 5, it will prove most transparent if the form (89) for the inner product is used.

Summing over all amounts m and positions $\{x_i\}$ of external particles, dividing with the norm, and furthermore dividing by $m!$ to kill permutations, we get for the partition function

$$Z_\Psi = \mathrm{Tr}_\Psi \left(\mathrm{e}^{-H/T} \right) = \sum_{m=0}^{\infty} \tfrac{1}{m!} \left(\tfrac{k}{4\pi} \right)^m \mathrm{e}^{-\frac{e^2 k}{4\pi T} m} \ \langle \mathcal{Z}_m \rangle_{G/G} \ , \tag{92}$$

where

$$\mathcal{Z}_m = \int [dB] \ e^{\frac{k}{4\pi} \int |B|^2} \left[\text{Tr} \int (B_- - J_-)(B_+ + J_+) \right]^m .$$

(93)

This expression needs renormalization due to the singularities arising from the delta-function norms. Keeping this caveat in our mind, we can commence with formally computing Z_Ψ. Performing the sum over m before integrating over B we get

$$Z_\Psi = \frac{1}{\mathcal{N}} \left\langle \exp \left\{ -\frac{k}{8\pi} \left(\coth \frac{e^2 k}{8\pi T} - 1 \right) \ \text{Tr} \int J_+ J_- \right\} \right\rangle_{G/G} ,$$

(94)

with $\mathcal{N}$ the normalization of Eq. (63). This is exactly the full finite temperature partition function (60). Accordingly, the states (82) saturate the thermal ensemble of topologically massive 2+1-dimensional gauge theory in the strong coupling limit, and we have found all the physical states that contribute in this limit.

On the other hand, the functional integrations in the partition function can be computed directly, using the normalization (91) for the physical states. Due to the delta-function normalizability, the contribution of each Ψ_m diverges as $\delta(0)^m$. To get a finite result, we have to employ the logarithmic scaling of the coupling constant found in Section 6. If the coupling is scaled as $\frac{e^2 k}{8\pi T} \sim \ln \Lambda/\mu$, the contributions of the delta-function norms are canceled. In terms of the rescaled e^2, the partition function evaluates to

$$Z = \sum_{m=0}^{\infty} \frac{1}{m!} \ e^{-\frac{e^2 (k+c_v)}{4\pi T} m} \ V^m \ \left\langle \prod_{i=1}^{m} \mathcal{X}_{\text{Ad}} \right\rangle_{G/G} ,$$

(95)

where V is the volume of the two-manifold Σ. Here we have taken into account the zero-point energy subtracted from the Hamiltonian by the normal ordering in Eq. (31), by using the fact that the corresponding short distance singularity in E^2 is known to give rise to the shift $k \to k + c_v$ in WZNW states [39].

The partition function, within a logarithmic distance from the topological theory, thus remains computable in terms of purely topological quantities, the Verlinde numbers.

8   Conclusions

We have considered the strong coupling limit of (2+1)-dimensional topologically massive gauge theory. The Hamiltonian is just $H = e^2 \int E^2$, and the vacuum sector, corresponding to infinite coupling, is the one of pure Chern-Simons theory. Furthermore, in the functional Scrödinger picture, the vacuum can be described in terms of a G/G WZNW theory, the partition function of which is the normalization function of the wave functional. It is well known in fact, that G/G WZNW partition functions reproduce dimensions of canonically quantized Chern-Simons Hilbert spaces in terms of the so-called Verlinde formula [28].

Polyakov loops in the finite temperature theory correspond to punctures on the surface. Polyakov loop correlators can thus be calculated as dimensions of Chern-Simons theories on the corresponding surfaces. These, on the other hand, are given by correlation functions of group characters in the G/G model. Using the

Verlinde formula, we can thus calculate all vacuum correlators of Polyakov loops in strongly coupled topologically massive gauge theory. These results confirm the spontaneous center-symmetry breaking characteristic of deconfining gauge theories without fundamental matter.

By using the heat-kernel, the finite temperature partition function can be evaluated, this takes into account low-lying excited states at strong (but not infinite) coupling. The finite temperature effective theory is a perturbed G/G model, where the "level" of the principal sigma-model term is perturbed away from the topological (and conformal) fixed point.

We computed the one-loop effective potential for the Polyakov loop operators. It was found that one gets finite results for thermodynamical quantities if e^2/T scales like a logarithm of the fundamental cut-off. This indicates the domain of reliability of the strong coupling approximation. The effects of the suppressed magnetic field are truly negligible as long as we stay within logarithmic distance from the infinite coupling fixed point.

The minima of the one-loop effective potential lay on the Z_N center symmetry, indicating that Z_N is broken and that the theory is in a deconfined phase.

Finally, we constructed the low-lying excited states. Using holomorphic factorization of G/G models, we found them to be non-abelian Landau levels built upon Chern-Simons vacua on Riemann spheres with punctures. Gauge invariant states consist of external particle insertions corresponding to the punctures, plus glue created by the dynamic gauge field. We showed that the heat-kernel partition function was saturated by these states, and evaluated the partition function. With the logarithmic scaling of the coupling constant, the finite temperature partition function reduces to a sum of Verlinde factors with different numbers of insertions, i.e. of zero temperature (vacuum) correlators of the Polyakov loops corresponding to static external particles.

Acknowledgements

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